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The Method of Lagrange Multiplier

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Introduction

- In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equation constraints

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & g_i(x) = 0, i = 1, 2, \cdots, m \end{aligned}$$

- It is named after the mathematician Joseph-Louis Lagrange

A Simple Case

- The basic idea is to convert a constrained problem into a form such that the derivative test of an unconstrained problem can still be applied
- In general, the Lagrangian (or Lagrangian function) is defined by

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

where λ_i is called the Lagrange multiplier

A Simple Case (Contd.)

- In order to find the maximum (or minimum) of function f subject to the equality constraints $g_i(x) = 0$, find the stationary points of $\mathcal{L}$ considered as a function of x and the Lagrange multiplier λ

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 0$$

- Equivalently,

$$\frac{\partial f(x)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x} = 0 \quad \text{and} \quad g_i(x) = 0, \forall i = 1, 2, \cdots, m$$

A Toy Exercise

- A rectangular box without a lid is to be made from $27m^2$ of cardboard. Find the maximum volume of such a box

A Formal Statement

Theorem (Lagrange Multiplier Theorem)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *be the objective function,* $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ *be the constraint function, both of which have continuous fist derivatives. Let* x^* *be an optimal solution to the following optimization problem*

$$
\begin{aligned}
\max \quad & f(x) \\
\text{s.t.} \quad & g(x) = 0
\end{aligned}
$$

such that $\mathrm{Rank}(\mathrm{D}g(x^*)) = m < n$ *for the matrix of partial derivatives* $[\mathrm{D}g(x^*)]_{j,k} = \frac{\partial g_j}{\partial x_k}$*. Then, there exist a unique Lagrange multiplier* $\lambda^* \in \mathbb{R}^m$ *such that*

$$\mathrm{D}f(x^*) = (\lambda^*)^T \mathrm{D}g(x^*)$$

A Formal Statement (Contd.)

- At any local maximum (or minimum) of the function evaluated under the equality constraints, if constraint qualification applies, then the gradient of the function (at that point) can be expressed as a linear combination of the gradients of the constraints (at that point), with the Lagrange multiplier serving as coefficients
- This is equivalent to saying that, any direction perpendicular to all gradients of the constraints is also perpendicular to the gradient of the function
- More details can be found at `https://en.wikipedia.org/wiki/Lagrange_multiplier`

Thanks!

Q & A